REMARKS ON A PAPER OF GERONIMO AND JOHNSON

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0.1. **Character–automorphic Hardy Spaces.** Let E be a finite union of (necessary non–degenerate) arcs on the unit circle $\mathbb{T}$. The domain $\overline{\mathbb{C}}\setminus E$ is conformally equivalent to the quotient of the unit disk by the action of a discrete group $\Gamma=\Gamma(E)$. Let $z:\mathbb{D}\to\overline{\mathbb{C}}\setminus E$ be a covering map, $z\circ\gamma=z$, $\forall\gamma\in\Gamma$. In what follows we assume the following normalization to be hold

$$z:(-1,1)\to(a_0,b_0)\subset\mathbb{T}\setminus E,$$

where (a_0,b_0) is a fixed gap, $\mathbb{T}\setminus E=\cup_{j=0}^{g}(a_j,b_j)$. In this case one can chose a fundamental domain $\mathfrak{F}$ and a system of generators $\{\gamma_j\}_{j=1}^{g}$ of Γ such that they are symmetric with respect to the complex conjugation:

$$\overline{\mathfrak{F}}=\mathfrak{F},\quad \overline{\gamma_j}=\gamma_j^{-1}.$$

Denote by $\zeta_0\in\mathfrak{F}$ the preimage of the origin, $z(\zeta_0)=0$, then $z(\overline{\zeta_0})=\infty$. Let $B(\zeta,\zeta_0)$ and $B(\zeta,\overline{\zeta_0})$ be the Green functions with $B(\overline{\zeta_0},\zeta_0)>0$ and $B(\zeta_0,\overline{\zeta_0})>0$. Then

$$z(\zeta)=e^{ic}\frac{B(\zeta,\zeta_0)}{B(\zeta,\overline{\zeta_0})}. \tag{1}$$

It is convenient to rotate (if necessary) the set E and to think that $c=0$. Note that $B(\zeta,\zeta_0)$ is a character–automorphic function

$$B(\gamma(\zeta),\zeta_0)=\mu(\gamma)B(\zeta,\zeta_0),\quad \gamma\in\Gamma,$$

with a certain $\mu\in\Gamma^*$. By (1)

$$B(\gamma(\zeta),\overline{\zeta_0})=\mu(\gamma)B(\zeta,\overline{\zeta_0}),\quad \gamma\in\Gamma.$$

Recall that the space $A_1^2(\alpha)$, $\alpha\in\Gamma^*$ is formed by functions of Smirnov class in $\mathbb{D}$ such that

$$f|[\gamma](\zeta):=\frac{f(\gamma(\zeta))}{\gamma_{21}\zeta+\gamma_{22}}=\alpha(\gamma)f(\zeta),\quad \gamma=\begin{bmatrix}\gamma_{11}&\gamma_{12}\\ \gamma_{21}&\gamma_{22}\end{bmatrix},$$

and

$$||f||^2:=\int_{\mathbb{T}/\Gamma}|f(t)|^2\,dm(t)<\infty.$$

We denote by $k^\alpha(\zeta,\zeta_0)$ the reproducing kernel of this space and put

$$K^\alpha(\zeta,\zeta_0):=\frac{k^\alpha(\zeta,\zeta_0)}{||k||}=\frac{k^\alpha(\zeta,\zeta_0)}{\sqrt{k^\alpha(\zeta_0,\zeta_0)}}.$$

Notice that in our case $f(\zeta)\in A_1^2(\alpha)$ implies $\overline{f(\bar\zeta)}\in A_1^2(\alpha)$ and therefore

$$K^\alpha(\zeta_0,\zeta_0)=K^\alpha(\overline{\zeta_0},\overline{\zeta_0}).$$

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0.2. **A recurrence relation for reproducing kernels.** We start with

Theorem 0.1. *Systems*

$$\{K^\alpha(\zeta,\zeta_0), B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\}$$

and

$$\{K^\alpha(\zeta,\overline{\zeta_0}), B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0)\}$$

form orthonormal bases in the two dimensional space spanned by $K^\alpha(\zeta,\zeta_0)$ *and* $K^\alpha(\zeta,\overline{\zeta_0})$. *Moreover*

$$\begin{aligned} K^\alpha(\zeta,\overline{\zeta_0}) = &\quad a(\alpha)K^\alpha(\zeta,\zeta_0)+\rho(\alpha)B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0}),\\ K^\alpha(\zeta,\zeta_0) = &\quad \overline{a(\alpha)}K^\alpha(\zeta,\overline{\zeta_0})+\rho(\alpha)B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\end{aligned} \tag{2}$$

where

$$a(\alpha)=a=\frac{K^\alpha(\zeta_0,\overline{\zeta_0})}{K^\alpha(\zeta,\zeta_0)},\quad \rho(\alpha)=\rho=\sqrt{1-|a|^2}.$$

Proof. Let us prove the first relation in (2). It is evident that the vectors $K^\alpha(\zeta,\zeta_0)$ and $B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})$ are orthogonal, normalized and orthogonal to all functions f from $A^2_1(\alpha)$ such that $f(\zeta_0)=f(\overline{\zeta_0})=0$, that is to functions that form orthogonal compliment to the vectors $K^\alpha(\zeta,\zeta_0)$ and $K^\alpha(\zeta,\overline{\zeta_0})$. Thus

$$K^\alpha(\zeta,\overline{\zeta_0})=c_1K^\alpha(\zeta,\zeta_0)+c_2B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0}).$$

Putting $\zeta=\zeta_0$ we get $c_1=a$. Due to orthogonality we have

$$1=|a|^2+|c_2|^2.$$

Now, put $\zeta=\overline{\zeta_0}$. Taking into account that $K^\alpha(\zeta_0,\overline{\zeta_0})=\overline{K^\alpha(\overline{\zeta_0},\zeta_0)}$ and $B(\overline{\zeta_0},\zeta_0)>0$ we prove that c_2 being positive is equal to $\sqrt{1-|a|^2}$.

Note that simultaneously we proved that

$$\rho=B(\overline{\zeta_0},\zeta_0)\frac{K^{\alpha\mu^{-1}}(\overline{\zeta_0},\overline{\zeta_0})}{K^\alpha(\overline{\zeta_0},\overline{\zeta_0})}.$$

□

Corollary 0.2. *A recurrence relation for reproducing kernels generated by the shift of* Γ^* *on the character* μ^{-1} *is of the form*

$$\begin{aligned} &B(\zeta,\zeta_0)\left[K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\quad -K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\right]\\ &\qquad=\left[K^\alpha(\zeta,\zeta_0),\quad -K^\alpha(\zeta,\overline{\zeta_0})\right]\frac{1}{\rho}\begin{bmatrix}1 & a\\ \bar a & 1\end{bmatrix}\begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}.\end{aligned} \tag{3}$$

Proof. We write

$$\begin{aligned} &B(\zeta,\zeta_0)\left[K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\quad -K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\right]\\ &\quad=\left[B(\zeta,\overline{\zeta_0})K^{\alpha\mu^{-1}}(\zeta,\zeta_0),\quad -B(\zeta,\zeta_0)K^{\alpha\mu^{-1}}(\zeta,\overline{\zeta_0})\right]\begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}.\end{aligned}$$

Then, use (2). □

Corollary 0.3. *Let*

$$s^\alpha(z) := \frac{K^\alpha(\zeta,\overline{\zeta_0})}{K^\alpha(\zeta,\zeta_0)} \tag{4}$$

Then the Schur parameters of the function $\tau s^\alpha(z)$, $\tau\in\mathbb{T}$, *are*

$$\{\tau a(\alpha\mu^{-n})\}_{n=0}^\infty.$$

Proof. Let us note that (3) implies

$$s^\alpha(z) = \frac{a(\alpha) + z s^{\alpha\mu^{-1}}(z)}{1+\overline{a(\alpha)} z s^{\alpha\mu^{-1}}(z)}.$$

Then we iterate this relation. Also, multiplication by $\tau\in\mathbb{T}$ of a Schur class function evidently leads to multiplication by τ of all Schur parameters. □

Remark. Let

$$M(z;\alpha,\tau) = \frac{1+z\tau s^\alpha(z)}{1-z\tau s^\alpha(z)}, \tag{5}$$

$(\alpha,\tau)\in\Gamma^*\times\mathbb{T}\simeq\mathbb{T}^{g+1}$. Then

$$M(z;\alpha,\tau) = \int \frac{t+z}{t-z} d\sigma(t;\alpha,\tau)$$

gives $g+1$ parametric family of probabilistic measures on the unit circle. Let us point out the normalization conditions for M: $M(0)=1, M(\infty)=-1$.

0.3. **Example (one–arc case).** In this case $\overline{\mathbb{C}}\setminus E\simeq\mathbb{D}$, Γ is trivial, and

$$z = \frac{B(\zeta,\zeta_0)}{B(\zeta,\overline{\zeta_0})} = \frac{\frac{\zeta-\zeta_0}{1-\zeta\overline{\zeta_0}}\overline{\left(\frac{\overline{\zeta_0}-\zeta_0}{1-\overline{\zeta_0}^2}\right)}}{\frac{\zeta-\overline{\zeta_0}}{1-\zeta\zeta_0}\overline{\left(\frac{\overline{\zeta_0}-\zeta_0}{1-\zeta_0^2}\right)}} = -\frac{\zeta-\zeta_0}{\zeta-\overline{\zeta_0}}\frac{1-\zeta\zeta_0}{1-\zeta\overline{\zeta_0}}\frac{1-\overline{\zeta_0}^2}{1-\zeta_0^2}. \tag{6}$$

That is

$$b_0 = z(1) = -\frac{1-\zeta_0}{1+\zeta_0}\frac{1+\overline{\zeta_0}}{1-\overline{\zeta_0}},$$

and $a_0=\overline{b_0}$. We can put $\zeta_0 = ir$, $0<r<1$. Then

$$b_0 = \left(\frac{2r}{1+r^2} + i\frac{1-r^2}{1+r^2}\right)^2 = e^{2i\theta},$$

where

$$\sin\theta = \frac{1-r^2}{1+r^2}, \quad \theta\in(0,\pi/2).$$

Further, for such z

$$s(z) = \frac{K(\zeta,\overline{\zeta_0})}{K(\zeta,\zeta_0)} = \frac{\frac{1}{1-\zeta\zeta_0}}{\frac{1}{1-\zeta\overline{\zeta_0}}} = \frac{1-\zeta\overline{\zeta_0}}{1-\zeta\zeta_0}.$$

Thus

$$a = s(0) = \frac{1-|\zeta_0|^2}{1-\zeta_0^2} = \frac{1-r^2}{1+r^2} = \sin\theta.$$

The Schur parameters of the function $s_\tau(z) = \tau s(z)$ are

$$s_\tau(z)\sim\{\tau\sin\theta,\tau\sin\theta,\tau\sin\theta...\}.$$

0.4. **Lemma on the reproducing kernel.** Let us map (the unit circle of) z–plane onto (the upper half–plane of) λ–plane in such a way that $a_0 \mapsto 1$, $z(0) \mapsto \infty$, $b_0 \mapsto -1$. In this way ($\zeta \mapsto z \mapsto \lambda$) we get the function $\lambda = \lambda(\zeta)$ such that

$$z = \frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})} \frac{\lambda - \lambda_0}{\lambda - \overline{\lambda_0}}, \quad \lambda_0 := \lambda(\zeta_0). \tag{7}$$

Lemma 0.4. *Let $k^\alpha(\zeta) = k^\alpha(\zeta, 0)$ and $B(\zeta) = B(\zeta, 0)$ be subject to the normalization $(\lambda B)(0) > 0$. Denote by μ_0 the character generated by B, i.e., $B \circ \gamma = \mu_0(\gamma) B$. Then*

$$k^\alpha(\zeta, \zeta_0) = (\lambda B)(0) \frac{\overline{k^\alpha(\zeta_0)} \frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta) k^{\alpha\mu_0}(0)} - \overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0) k^{\alpha\mu_0}(0)}} k^\alpha(\zeta)}{\lambda - \overline{\lambda_0}}. \tag{8}$$

Proof. We start with the evident orthogonal decomposition

$$A_1^2(\alpha\mu_0) = \{k^{\alpha\mu_0}\} \oplus B A_1^2(\alpha).$$

We use this decomposition to obtain

$$\lambda B f = (\lambda B)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta)}{k^{\alpha\mu_0}(0)} + B\tilde{f}, \quad \tilde{f} \in A_1^2(\alpha).$$

Dividing by B and using the orthogonality of the summands, we get

$$P_+(\alpha)\lambda f = \tilde{f} = \lambda f - (\lambda B)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta) k^{\alpha\mu_0}(0)}, \tag{9}$$

where $P_+(\alpha)$ is the orthoprojector onto $A_1^2(\alpha)$.

Thus, on the one hand, for arbitrary $f \in A_1^2(\alpha)$

$$\langle (\lambda - \lambda_0) f, k^\alpha(\zeta, \zeta_0) \rangle = \{P_+(\alpha)(\lambda - \lambda_0) f\}(\zeta_0). \tag{10}$$

By virtue of (9) we have

$$\begin{aligned} \{P_+(\alpha)(\lambda - \lambda_0) f\}(\zeta_0) &= \lambda(\zeta_0) f(\zeta_0) - (B\lambda)(0) f(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0) k^{\alpha\mu_0}(0)} \\ -\lambda_0 f(\zeta_0) &= -(B\lambda)(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0) k^{\alpha\mu_0}(0)} \langle f, k^\alpha \rangle. \end{aligned} \tag{11}$$

On the other hand, since the function λ is real on $\mathbb{T}$,

$$\begin{aligned} \langle (\lambda - \lambda_0) f, k^\alpha(\zeta, \zeta_0) \rangle &= \langle f, (\lambda - \overline{\lambda_0}) k^\alpha(\zeta, \zeta_0) \rangle \\ &= \langle f, P_+(\alpha)(\lambda - \overline{\lambda_0}) k^\alpha(\zeta, \zeta_0) \rangle. \end{aligned} \tag{12}$$

Comparing (10) and (11) with (12), we get

$$P_+(\alpha)(\lambda - \overline{\lambda_0}) k^\alpha(\zeta, \zeta_0) = -\overline{(B\lambda)(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0) k^{\alpha\mu_0}(0)}} k^\alpha(\zeta).$$

Using (9) again, we get

$$\begin{aligned} (\lambda - \overline{\lambda_0}) k^\alpha(\zeta, \zeta_0) - (B\lambda)(0) k^\alpha(0, \zeta_0) &\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta) k^{\alpha\mu_0}(0)} \\ &= -\overline{(B\lambda)(0) \frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0) k^{\alpha\mu_0}(0)}} k^\alpha(\zeta). \end{aligned}$$

Since $k^\alpha(0,\zeta_0)=\overline{k^\alpha(\zeta_0)}$, we have

$$
\begin{aligned}
&(\lambda-\overline{\lambda_0})k^\alpha(\zeta,\zeta_0)\\
&=(B\lambda)(0)\left\{\overline{k^\alpha(\zeta_0)}\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)k^{\alpha\mu_0}(0)}-\overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)k^{\alpha\mu_0}(0)}}k^\alpha(\zeta)\right\}.
\end{aligned}
$$

The lemma is proved. □

Corollary 0.5. *In the introduced above notations*

$$
\begin{aligned}
zs^\alpha(z)&=\frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})}\frac{\lambda-\lambda_0}{\lambda-\overline{\lambda_0}}\frac{K(\zeta,\overline{\zeta_0})}{K(\zeta,\zeta_0)}\\
&=\frac{B(0,\zeta_0)}{B(0,\overline{\zeta_0})}\frac{k^\alpha(\zeta_0)\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)}-\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)}k^\alpha(\zeta)}{\overline{k^\alpha(\zeta_0)}\frac{k^{\alpha\mu_0}(\zeta)}{B(\zeta)}-\overline{\frac{k^{\alpha\mu_0}(\zeta_0)}{B(\zeta_0)}}k^\alpha(\zeta)}\\
&=\frac{B(0,\zeta_0)k^\alpha(\zeta_0,0)}{B(0,\overline{\zeta_0})k^\alpha(0,\zeta_0)}\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}},
\end{aligned}
\tag{13}
$$

where

$$
r(\lambda;\alpha):=\frac{(\lambda B)(0)}{B(\zeta)}\frac{k^\alpha(0)}{k^{\alpha\mu_0}(0)}\frac{k^{\alpha\mu_0}(\zeta)}{k^\alpha(\zeta)}.
\tag{14}
$$

Let us point out that functions (14) are important in the spectral theory of Jacobi matrices [Sodin–Yuditskii], they are normalized by

$$
r(\lambda;\alpha)=\lambda+...,\quad \lambda\to\infty.
$$

Corollary 0.6. *Let*

$$
\tau(\alpha)=\left\{\frac{B(0,\zeta_0)k^\alpha(\zeta_0,0)}{B(0,\overline{\zeta_0})k^\alpha(0,\zeta_0)}\right\}^{-1}.
$$

Then

$$
M(z;\alpha,\tau(\alpha))=\frac{r(\lambda;\alpha)-\Re r(\lambda_0;\alpha)}{i\Im r(\lambda_0;\alpha)}.
\tag{15}
$$

Proof. By definition (5) and (13)

$$
M(z;\alpha,\tau(\alpha))=\frac{1+\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}}}{1-\frac{r(\lambda;\alpha)-r(\lambda_0;\alpha)}{r(\lambda;\alpha)-\overline{r(\lambda_0;\alpha)}}}=\frac{r(\lambda;\alpha)-\Re r(\lambda_0;\alpha)}{i\Im r(\lambda_0;\alpha)}.
$$

□

0.5. **Main Theorem.** Let E be a finite union of arcs on the unit circle T, $\mathbb{T}\setminus E=\cup_{j=0}^g(a_j,b_j)$, normalized by the condition $c=0$ in (1). Let $\mathfrak{R}(E)$ be the hyper–elliptic Riemann surface with ramification points $\{a_j,b_j\}_{j=0}^g$ (double of the domain $\overline{\mathbb{C}}\setminus E$). Let us introduce a special collection of divisors on $\mathfrak{R}(E)$:

$$
D(E)=\{D=\sum_{j=0}^g(t_j,\epsilon_j):t_j\in[a_j,b_j],\quad \epsilon_j=\pm1\},
$$

where $(t_j,1)$ (correspondently $(t_j,-1)$) denotes a point on the upper (lower) sheet of the double $\mathfrak{R}(E)$, naturally, $(a_j,1)\equiv(a_j,-1)$ and $(b_j,1)\equiv(b_j,-1)$. Note that topologically $D(E)$ is the torus $\mathbb{T}^{g+1}$.

Following [Akhiezer–Tomchuk, Pehersorfer–Steinbauer, Geronimo–Johnson], we consider the collection of functions

$$\mathfrak{M}(E) = \{M(z, D) : D \in D(E)\}$$

given in $\overline{\mathbb{C}} \setminus E$ such that $M(z, D)$ can be extended on $\mathfrak{R}(E)$ as a rational function on it that has exactly D as the divisor of poles and meets the normalizations $M(0, D) = 1$, $M(\infty, D) = -1$. Note that the function is uniquely defined by D and the normalizations and has the integral representation

$$M(z, D) = \int \frac{t+z}{t-z}\, d\sigma_D(t),$$

with a probabilistic measure σ_D on $\mathbb{T}$.

Theorem 0.7. *A given $D \in D(E)$ there exists a unique $(\alpha, \tau) \in \Gamma^* \times \mathbb{T}$ such that the reflection coefficients related to the orthogonal polynomials with respect to σ_D are $\{\tau a(\alpha\mu^{-n})\}_{n=0}^{\infty}$.*

Proof. We only have to show that a given $M(z) = M(z, D)$ is of the form (5) with a certain (α, τ) and to use Corollary 0.3.

First, in the collection of functions

$$M_\theta(z) = \frac{\cos\frac{\theta}{2} M(z) - i\sin\frac{\theta}{2}}{-i\sin\frac{\theta}{2} M(z) + \cos\frac{\theta}{2}}, \quad 0 \le \theta < 2\pi, \tag{16}$$

chose that one that has a pole at $z(0) \in (a_0, b_0)$. Since $M(z) \in i\mathbb{R}$ when $z \in (a_0, b_0)$, there exists a unique θ that satisfied this condition. It is important, that $M_\theta \in \mathfrak{M}(E)$, that is there exists a unique D_θ such that $M_\theta(z) = M(z, D_\theta)$.

Let us denote by $\tilde{\mathfrak{R}}(E)$ the Riemann surface that we obtain by cutting and glueing two copies of the λ–plane (see (7)) and by $\tilde{D}$ the divisor on $\tilde{\mathfrak{R}}(E)$ that corresponds to a divisor $D \in D(E)$. As it well known (see e.g. [Sodin-Yuditskii]), given $\tilde{D}_\theta$ there exists a unique $\alpha \in \Gamma^*$ such that $\tilde{D}_\theta$ is the divisor of poles of a function of the form (14).

Now, consider the function

$$M(z; \alpha, \tau(\alpha)) = \frac{r(\lambda(z); \alpha) - \Re r(\lambda_0; \alpha)}{i\Im r(\lambda_0; \alpha)}$$

with the chosen α. It belongs to the class $\mathfrak{M}(E)$ and, according to its definition, has D_θ as the divisor of poles. Therefore, by uniqueness,

$$M_\theta(z) = M(z; \alpha, \tau(\alpha)). \tag{17}$$

Substituting (17) in (16) and solving for $M(z)$, we get

$$M(z) = M(z; \alpha, \tau(\alpha) e^{-i\theta}).$$

The theorem is proved. □